MATHEMATICIANS' EXAMPLE-RELATED ACTIVITY IN FORMULATING CONJECTURES

Elise Lockwood[1], Alison G. Lynch[2], Amy B. Ellis[2], Eric Knuth[2]

[1]Oregon State University, [2]UW-Madison

This paper explores the role examples play in mathematicians' conjecturing activity. While previous research has examined example-related activity during the act of proving, little is known about how examples arise during the formulation of conjectures. Thirteen mathematicians were interviewed as they explored tasks that required the development of conjectures. During the interviews, mathematicians productively used examples as they formulated conjectures, particularly by creating systematic lists of examples that they examined for patterns. The results suggest pedagogical implications for explicitly targeting examples in conjecturing, and the study contributes to a body of literature that points to the benefits of exploring, identifying, and leveraging examples in proof-related activity.

INTRODUCTION AND MOTIVATION

Proof is a crucial aspect of mathematical practice, and researchers have emphasized its importance in the mathematics education of students across grade levels (e.g., Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002; Sowder & Harel, 1998). However, there is much evidence that students at all levels struggle with learning to prove (e.g., Healy & Hoyles, 2000; Kloosterman & Lester, 2004; Knuth, Choppin, & Bieda, 2009; Porteous, 1990). One way to gain insight into how better to help students is to study the work of mathematicians, who are themselves successful at proof. Indeed, there is a history of research that studies mathematicians' thinking and leveraging those findings for possible pedagogical implications for students (e.g., Carlson & Bloom, 2005; Weber, 2008). Thus, given the essential role examples play in mathematicians' proof-related activities (e.g., Epstein & Levy, 1995), we examine mathematicians' work on conjectures and draw potential pedagogical insights. In this paper, we continue our previous work with mathematicians (Lockwood, Ellis, Dogan, Williams & Knuth, 2012; Lockwood, Ellis, & Knuth, 2013) by studying mathematicians' example-related activity as they engage in formulating conjectures. Our examination details the ways in which mathematicians systematically generated and used examples in developing conjectures and discusses implications for the teaching and learning of proof.

RELEVANT LITERATURE AND THEORETICAL PERSPECTIVE

In this paper, we follow Bills and Watson's (2008) lead by defining an example as "any mathematics object from which it is expected to generalize" (p. 78). In defining *proof*, we draw on Harel and Sowder's (1998) definition, which is "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241).

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 129-136. Vancouver, Canada: PME.

Harel and Sowder further distinguish between two kinds of activity associated with proving – *ascertaining* (removing one's own doubts) and *persuading* (removing others' doubts) (p. 241).

While much of the literature emphasizes limitations of example-based reasoning (particularly as a means of justification), a number of researchers have suggested the potential value examples may play in proof-related activity. As Epstein and Levy (1995) note, "Most mathematicians spend a lot of time thinking about and analyzing particular examples….It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Likewise, Harel (2008) notes that, "Examples and non-examples can help to generate ideas or give insight [about the development of proofs]" (p. 7). Other researchers have similarly reported that students and mathematicians display strategic uses of examples that benefit their proof-related activities (e.g., Ellis, et al., 2012; Garuti, Boero & Lemut, 1998; Pedemonte, 2007; Sandefur, et al., 2013; Weber, 2008). Our work builds upon such studies by seeking to identify potentially fruitful aspects of example-related activity in the development and proving of conjectures.

The study presented in this paper is situated within a framework developed by Lockwood, et al. (2012) and refined in Lockwood, et al. (2013) that categorizes *example types, example uses*, and *example strategies*. While the framework is not presented here due to space, it served as a broader context that guided data analysis.

METHODS

We conducted hour-long interviews with mathematicians in which they were presented with one or two mathematics tasks. A member of the research team (an advanced mathematics PhD student) conducted the interviews and participated in the analysis. During the interviews, the mathematicians were given time to work on the tasks on their own and were asked to think aloud; generally, the interviewer did not interrupt except to ask clarifying questions or to answer questions from the mathematicians. The mathematicians used Livescribe pens during the interviews, pens that both audio-record and keep live records of the mathematicians' written work. This technology allows for efficient data collection and facilitates rich analysis by providing both audio and written work of the interviews that can be re-played in real time, with the audio synced with the written work.

Participants

The participants were thirteen mathematicians from a large Midwestern university. The participants included seven professors, three postdocs, and three lecturers, with eight males and five females. Twelve participants hold a Ph.D. in mathematics, and one participant holds a Ph.D. in computer science. There were a variety of mathematical areas represented, including topology, number theory, and analysis.

Tasks

All thirteen mathematicians worked on the *Interesting Numbers* task while seven also tried an additional task that is not reported here. The *Interesting Numbers* task states, "Most positive integers can be expressed with the sum of two or more consecutive integers. For example, 24 = 7 + 8 + 9, and 51 = 25 + 26. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore interesting. What are all the interesting numbers?" One approach to solving this task is as follows: It can be shown that the sum of any two or more consecutive positive integers has an odd factor greater than 1. Conversely, if a positive integer N has an odd factor $k > 1$, it can be shown that N can be written as the sum of either k or $2N/k$ consecutive positive integers, whichever is smaller. The interesting numbers are thus exactly those positive integers that have no odd factors greater than 1. In other words, the interesting numbers are the powers of 2.

Both tasks were chosen because: a) they were accessible (i.e., did not require specialized content knowledge and were easy to explore) but were not trivial (i.e., a solution was not immediately available), b) they were accessible to the interviewer, allowing her to ask relevant questions and engage with the mathematicians, and c) they involved open-ended questions that would facilitate conjecturing. These were not "prove or disprove" statements that already stated a conjecture, but rather these tasks required that certain numbers and sets be characterized. Through such activity, the mathematicians developed conjectures that they could then attempt to prove.

Analysis

As mentioned, the Livescribe pen yields both an audio record of the interview and a pdf document of the interviewee's written work (synced with the audio). In this pdf, the audio and the written work can be played back, so the researcher can see and hear what was written and said in real time. The interviews were also transcribed. To analyse these interviews, two members of the research team independently coded and then discussed four interviews using Lockwood, et al.'s (2012, 2013) framework for example types, uses, and strategies. In coding the interviews, the researchers also noted codes that emerged from their analysis and that were not captured by the previously developed framework. After the four interviews were initially coded, compared, and discussed, the remaining nine interviews were split up and coded. The two researchers came together regularly to discuss any issues or questions that arose in analysing these remaining interviews. After completing the coding of all the interviews, the researchers met to discuss phenomena and themes that pertained especially to conjecturing and revisited relevant episodes in the transcripts.

RESULTS

While we had previously (Lockwood, et al., 2013) reported on how mathematicians generated and used examples as they *proved*, here we report on their work with examples as they *conjectured*. In this section we elaborate a key phenomenon that we

observed as mathematicians used examples while formulating conjectures. We call this phenomenon "Data Collection," in which the mathematicians systematically and, in some sense exhaustively, went through every example in a finite sequence in order to gather information. The mathematicians generated examples based on sequentially exhausting a small list of examples, and they then subsequently reflected back on these organized example lists in order to formulate a conjecture. This activity was productive for some mathematicians, as we explore below, suggesting that there is potential value in the methodical generation of examples in formulating conjectures.

To illustrate this phenomenon, we present Mathematician 1's (M1 – a professor) work on the *Interesting Numbers* task. M1 began by computing a sequence of small sums: 1+2=3, 2+3=5, 3+4=7, and 4+5=9. From these examples, he recognized that odd numbers greater than 1 could not be interesting. He proved this fact algebraically by showing that any odd number $2n+1$ is the sum of n and $n+1$. Continuing with algebra, he then looked at general sums of 3, 4, and 5 consecutive numbers beginning with n. Each case gave him an algebraic expression ($3n+3$, $4n+6$, $5n+10$) representing numbers that were not interesting, from which he tried to generalize.

After some time, M1 recognized that his algebraic manipulation had not illuminated a conjecture, and he said, "Okay. So at this point, I would start over and try and do something a little more visual." He then drew a number line and began to write out the numbers. Because M1 already knew that the odd numbers were not interesting, he crossed those out as he wrote. He then proceeded to go through the even numbers and cross out those of the form $3n+3$, $4n+6$, and $5n+10$ for some n (Figure 1). After working through the numbers 1 through 21, he concluded, "well, the answer does kind of pop out that it's the powers of 2, doesn't it?" By actually writing out the examples and then crossing out non-interesting numbers, the pattern of numbers not crossed out – 1, 2, 4, 8, and 16 – stood out in his figure. His construction of the complete table, and his subsequent reflection on it, suggest the "Data Collection" phenomenon – he systematically gathered a complete sequence of examples and deduced patterns from them.

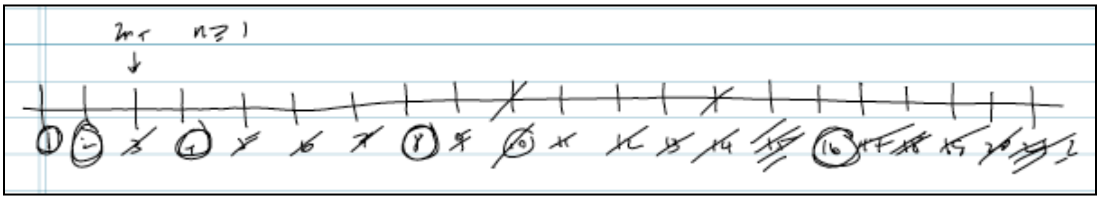

Figure 1: The "visual" list from which the powers of 2 conjecture emerges

We perceive that M1's prior knowledge and experience made him attuned to this sequence of numbers as powers of 2. M1 continued to pursue the powers of 2, saying, "Okay, so, um, so at this point I would maybe try the next one, 32," and he proceeded to write a conjecture that interesting numbers are powers of 2. To us, M1's careful construction of examples allowed for a common, familiar pattern to emerge visually on the page. M1's work suggests that the methodical generation of examples (what we call Data Collection) facilitated the efficient formulation of the conjecture.

As another, perhaps more extreme, example, we see in Figure 2 a table that M10 (a professor) created. This displays a great deal of care in detailing out a large number of cases. He also demonstrated Data Collection and formulated the correct conjecture by making note of the numbers that were not in the table.

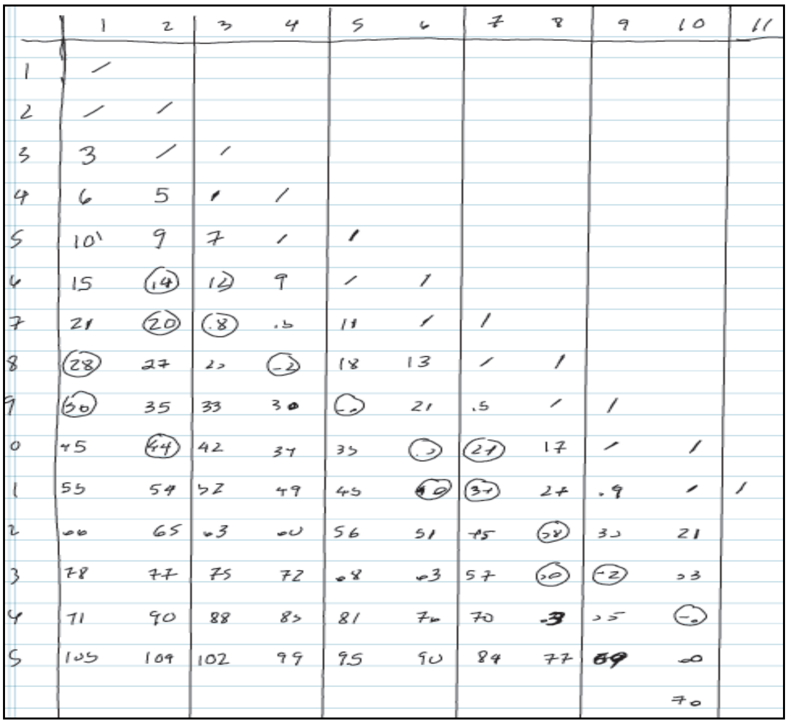

Figure 2: M10's table

To see why the phenomenon of data collection was especially useful, we note that not all mathematicians engaged with examples in this way. In contrast to M1's work, another mathematician, M6, did not generate data and detect a pattern. Instead, M6 developed an algebraic expression for a general non-interesting number, written as the sum of *n* consecutive integers starting with *k*. Starting with an arbitrary number (represented by $2^p \cdot q$ with *q* odd), M6 tried to find *k*, *n* (in terms of *p*, *q*) to make $2^p \cdot q$ non-interesting. Using only algebraic manipulation, M6 eventually found that this could be done if and only if $q > 1$. This result yielded the correct conjecture, but it took him more than twice as long (38 minutes) to find than the average time among the mathematicians that generated data (16 minutes). While the algebraic exploration was not an incorrect approach, we suspect that for conjecturing purposes, it did not so clearly illuminate potential patterns as the actual generation of concrete examples did. Indeed, unlike M1's work, in which the powers of 2 conjecture fell out almost immediately upon exploring examples, the pattern of the interesting numbers was obfuscated for M6 by the algebraic manipulation.

In addition to helping mathematicians formulate a correct conjecture, we present two ways in which Data Collection was efficacious in supporting mathematicians' conjecturing: Lemma Development, and Preliminary Conjecture Breaking. First,

observations from generated data lead to lemmas, which in turn informed the development of conjectures. For example, M3 first looked at the numbers 1 to 14 and tried to write each one as a sum of consecutive numbers. He noticed that odd numbers were sums of 2 consecutive numbers, multiples of 6 were sums of 3 consecutive numbers, and numbers congruent to 2 mod 4 other than 2 were sums of 4 consecutive numbers. From these observations, M3 proved lemmas stating that these types of numbers were non-interesting. These lemmas allowed M3 to restrict his attention to multiples of 4, which led to the development of the full conjecture.

Second, the data collection also allowed the mathematicians to find examples that broke preliminary conjectures, which in turn led to the articulation of more accurate conjectures. This is seen in M4, who initially conjectured that the interesting numbers were the non-primes after looking at the numbers 1 to 6 (and incorrectly deciding that 6 was interesting). He continued on to look at the numbers 7 to 10 before he realized his mistake, saying about 6, "Oh, 1, 2... 1 plus 2 plus 3. Right. Revise conjecture. So far, so, the interesting numbers so far are 4, 8, [...] It looks like it's the [multiples] of 4." M4 revised his conjecture once more (to a correct conjecture) when he looked at 11, 12 and 13 and discovered that 12 was also not interesting.

DISCUSSION AND CONCLUSIONS

The results highlight ways in which specific example-related activity like Data Collection may play a valuable role in the development of conjectures. In this section, we discuss three aspects of the results and suggest potential implications for students. First, some mathematicians (as seen in M1 and M10) took the time painstakingly to catalogue a number of examples. The generation of sequences of examples and subsequent reflection on them enabled the mathematicians to formulate conjectures effectively and efficiently. Students may thus benefit from generating comprehensive sets of data that they can survey in search of patterns, which in turn could illuminate conjectures. It is important to emphasize for students that such work may take patience and care. Second, also notable is the fact that these mathematicians engaged in deliberate and strategic example generation, which stands in contrast to less systematic behaviour often found in students' work with examples. For students, then, there might be value in helping them learn to be more strategic and methodical in their use of examples, going beyond finding a few confirming examples that simply come to mind. Third, in some of the mathematicians (such as M6) we saw an immediate application of algebraic techniques that were less efficacious for conjecturing than the Data Collection was. We suspect that some students may put a premium on algebraic techniques and may assume that algebraic activity is more sophisticated than generating examples. Our findings suggest that students should be encouraged to engage with and see the value in finding concrete examples when conjecturing and not simply to apply algebraic formulas and techniques. As a final point of discussion, we note that the tasks in our study were well suited to facilitate Data Collection. Other tasks might be more or less effective in fostering conjecturing. Instructors should be

aware of what kinds of activity and thinking certain tasks elicit and should expose students to tasks that might encourage Data Collection activities.

In this paper, we have reported on a beneficial phenomenon that emerged when mathematicians used examples during the activity of mathematical conjecturing. In this Data Collection phenomenon, mathematicians generated sequential lists of examples and used these lists in order to find patterns that might lead to conjectures. We saw this activity help mathematicians formulate correct conjectures, but it also helped with developing useful lemmas and also breaking initial conjectures to arrive at more accurate ones. These findings contribute to work that has been done previously that highlights the role of examples in mathematicians' proving, and it adds to the overall narrative that examples play a vital role in mathematicians' proof-related activity. The productive ways in which mathematicians use examples in formulating conjectures provide interesting and much-needed insights into proving and conjecturing into K-16 mathematics education.

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